Microeconomics notes

Pietro Battiston

April 3, 2011

These are the notes I took during the DEFAP [1] course of Microeconomics I, thought by prof. Nielsen.

01/13/2011

Microeconomics is an approach focusing on individual behaviour. In this first part of the sequence we will focus on the consumer's demand.

We must first consider a fundamental object: a *commodity*, which we will assume is well defined in some measurement unit.

Our *commodity space* will be $\mathbb{R}^L$ - for L commodities. In the future, we may have to do with an infinite number of commodities, but not in this course.

We have a commodity vector:

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_L \end{pmatrix}.$$

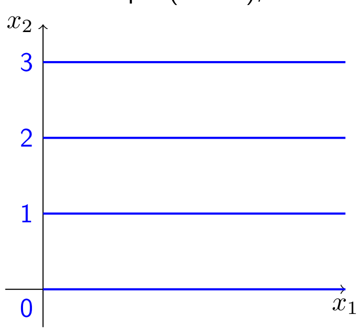
When we talk about the consumer, we have to talk about his *consumption set*, which is very likely to change from consumer to consumer, as well as his *preferences*. The theory is "general" in that by changing the form of the different objects, it should be able to describe *any* consumer.

Still, the validity of it will be judged based on its *relevance*, even more than correctness.

The consumption set is typically:

$$X \subset \mathbb{R}^L_+.$$

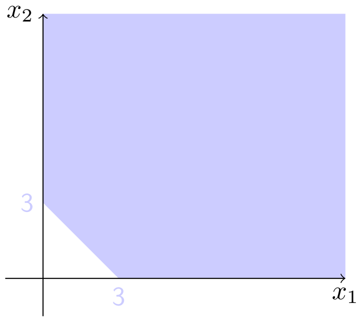
As an example ($L = 2$),



[1] http://scuoledidottorato.unicatt.it/defap

in this graph, the consumer can consume any amount of good 1, but only integer amounts of good 2.

Another example:



(this may be not particularly relevant in our advanced economies, but may be better suited to a developing economy where the set of food allocations avoiding starvation is a sensible object).

We will usually consider

$$\mathcal{X} = \mathbb{R}^L_+ - \left\{x \in \mathbb{R}^L | X_l \geq 0 \, \forall l\right\}$$

thought that same consumption set will seem awkward in some cases.

We will often distinguish the *consumption set* from other restrictions, typically depending from wage and prices. The combination of all restrictions and of preferences will yield the *consumption vector*.

0.1 Preference relation

We will indicate the preference relation as $\succeq$. We read $X \succeq Y$ as "X is at least as good as Y", or is "weakly preferred".

By $\sim$ we instead mean equivalence - $x \sim y$ if the consumer is indifferent between the two alternatives.

Basic assumptions on preferences:

1. *completeness*: $\forall x, y \in \mathcal{X}$ either $x \succ y$ or $y \succeq x$,
2. *transitivity*: $x \succeq y, y \succeq z \implies x \succeq z$.

The main point of transitivity is that if we get a guy whose preferences do not respect it, we can make money just selling him the goods, exchanging with those he has in the right way.

That's why we call preferences *rational* if they respect the two assumptions seen so far.

We now move to less fundamental - and more controversial - assumptions. For instance:

- *(A) monotonicity*: $x >> y$ (which means "for every component i of the vectors, $x_i > y_i$"[2]), implies $x \succ y$;
- *(B) strict monotonicity*: $x \geq y \wedge x \neq y \implies x \succ y$.

[2]We will use the symbol $\geq$ to say "$>>$ or $=$".

This can make sense, by the way, only if all commodities are *goods* - and that can be seen as a definition of what a "good" is.

There are weaker notions of monotonicity: for instance, *local non-satiation*:

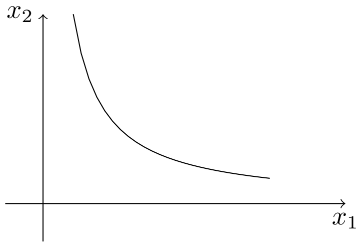
$$\forall x \in \mathcal{X},\ \forall \varepsilon > 0,\ \exists y : y \succ x \wedge \|x - y\| < \varepsilon.$$

Though weaker, it is affected basically by the same possible criticisms than the two kinds of monotonicity: there may be some "ideal" allocation from which we just have no desire to move away. One defense is that if we *had* reached this satiation point, then we would not talk at all about economics - the study of the question "can we allocate goods better?". So from the point of view of economics, this is a very relevant assumption.

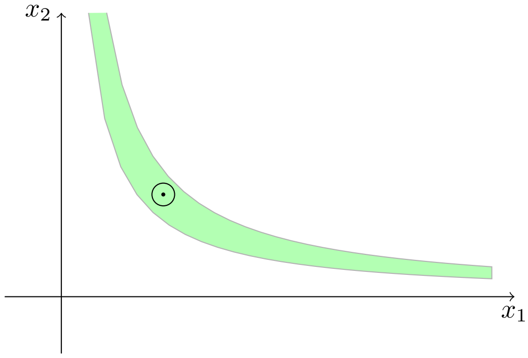
0.2 Indifference sets

Given some point $x \in \mathcal{X}$, the indifference set at x is $\{Y \in \mathcal{X} : x \sim y\}$.

A typical example is an indifference *curve*:



which is a restriction which typically makes sense, since if we assume it is "thick":



then we can find some point with some circle around it which goes against the assumption of local non-satiation.

We can now define *upper contour sets* at some point x as:

$$UCS(x) = \{y \in \mathcal{X} : y \succeq x\}$$

and *lower contour sets* as

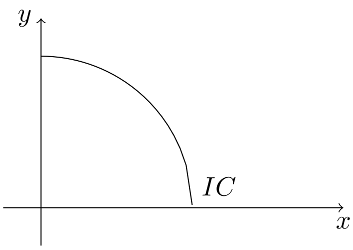
$$LCS(x) = \{y \in \mathcal{X} : X \succeq y\}.$$

Now comes a highly problematic assumption: *convexity*. It *is* problematic, but also very useful. We assume that $\succeq$ is convex, that is, that the upper contours set for any $x \in \mathcal{X}$ is convex.

Formally:

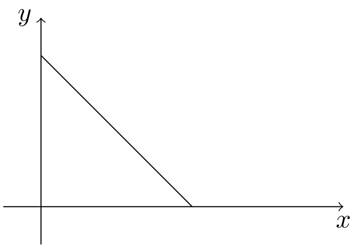
$$\forall x,\ \forall y, z \in UCS(x)\ \forall \alpha \in [0,1]\ \alpha y + (1-\alpha) z \in UCS(x).$$

Here's an example of a *non-convex* preference (we draw the indifference curve):



The economic interpretation of convexity is that *consumers do not appreciate extremes*: if we like the same 5 bananas or 7 apples, we prefer to both options a linear combination of the two.

The convexity assumption does not rule out preferences looking like



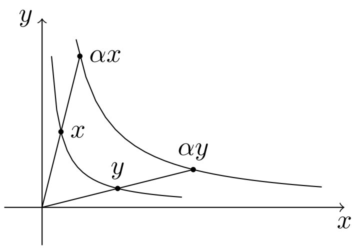
which however are ruled out by a stronger assumption: *strict* convexity:

$$\forall x,\ \forall y, z \in UCS(x), y \neq z, \alpha \in (0,1),\ \alpha y + (1-\alpha) z \succ x.$$

0.3 Further assumptions

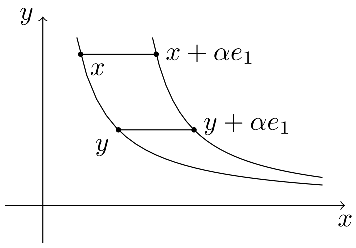
Two more assumptions often found in the literature are the ones of:

- *omothetic preferences*: $x \sim y \wedge \alpha > 0 \implies \alpha x \sim \alpha y$;



- *quasilinear preferences*: commodity 1 is called *numerator good*, and

$$x \sim y \iff x + \alpha e_1 \sim y + \alpha e_1\ \forall \alpha.$$



0.4 Utility

An *utility representation* of $\succeq$ is

$$\mathcal{U} : \mathcal{X} \to \mathbb{R} : \qquad \mathcal{U}(x) \geq \mathcal{U}(y) \iff x \succeq y.$$

Not all preferences can be represented. For instance, if a preference can be represented, then it is rational, since the complete ordering that is naturally given on $\mathbb{R}$ translates to a complete ordering on $\mathcal{X}$ (given x and y, either $\mathcal{U}(x) \geq \mathcal{U}(y) \implies x \succeq y$ or the converse is true).

A similar argument shows that any representable preference is transitive (again, transitivity holds on $\mathbb{R}$).

Still, not all rational preferences can be represented with an utility - a classical counterexample being *lexicographic* preferences:

$$x \succeq y \stackrel{\text{def}}{\iff} x_1 \geq y_1 \vee (x_1 = y_1 \wedge x_2 \geq y_2).$$

0.5 Continuity

A new assumption on preferences will guarantee us their representability - *continuity*:[3]

$$\forall \{x_n\}, \{y_n\}, \qquad x_n \to x, \; y_n \to y, \; x_n \succeq y_n \forall n \implies x \succeq y.$$

This is sort of a *technical* assumption - doesn't have a particularly important economic meaning... apart from the fact that lexicographic preferences do not respect it.

An analogous way of expressing it is that all $UCS(x)$ and $LCS(x)$ must be closed. It is easy to see the equivalence. For instance, if we take $x_n = x$, the conditions says that

$$y_n \succeq x_n = x \; \forall n$$

$$\Updownarrow \text{def}$$

$$y_n \in UCS(x) \; \forall n \implies y \in UCS(x)$$

and this is precisely the definition of $UCS(x)$ being closed.

Theorem 1. *If $\succeq$ is rational and continuous, then it has a utility representation $\mathcal{U} : \mathcal{X} \to \mathbb{R}$.*

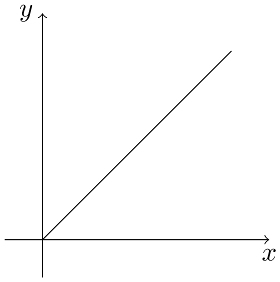
Under the further assumption $\mathcal{X} = \mathbb{R}^L$ and $\succeq$ monotonic, we have that $\mathcal{U}$ is also monotonic.

[3]By $\{x_n\}$, or $\{x_n\}_{n=1}^{\infty}$ we denote a sequence $\{x_1, x_2, x_3, \dots\}$.

Proof. We show the stronger (and easier to prove) version, assuming monotonicity. Let

$$e = (1, 1, \dots, 1);$$

we want to show that for each bundle x there is a unique number $\alpha(x)$ such that $e \sim \alpha x$.



First, we can show that $x \succeq 0\ \forall x \in \mathbb{R}_+^L$ (which is quite intuitive, given monotonicity). Then, we consider $\overline{\alpha}$ such that $\overline{\alpha} e \succeq x$. For instance, we can choose

$$\overline{\alpha} = \max\{x_1, x_2, \dots, x_L\} + 1.$$

We know that $0 \in LCS(x)$ and $\alpha e \in UCS(x)$ - they are both non empty.

That means that if we define

$$\overline{A} = \{\alpha \geq 0 : \alpha e \succeq x\}$$
$$\underline{A} = \{\alpha \geq 0 : x \succeq \alpha e\}$$

we can write $\overline{A} \neq \emptyset \neq \underline{A}$.

Now, completeness tells us that for any $\alpha \in \mathbb{R}_+$, either $\alpha e \geq x$ or $x \geq \alpha e$. So $\overline{A} \cup \underline{A} = \mathbb{R}_+$.

We hence know that $\overline{A} \cap \underline{A} \neq \emptyset$, because $\mathbb{R}_+^L$ is a *connected* set and the two sets are closed (continuity guarantees that). So we have some intersection: it is $\min \overline{A} = \max \underline{A} = \alpha(x)$.

The second step consists in showing that $\alpha(x)$ represents $\succeq$, that is if $x \succeq y$, then $\alpha(x) \geq \alpha(y)$. This is easy:

$$\alpha(x)e \sim x \succeq y \sim \alpha(y)e \implies \alpha(x) \geq \alpha(y).$$

and vice versa:

$$\alpha(x) \geq \alpha(y) \implies x \sim \alpha(x)e \succeq \alpha(y)e \sim y,$$

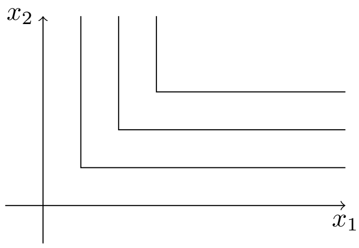
So finally, $\mathcal{U}(x) = \alpha(x)$.

The proof of continuity of $\mathcal{U}$ is missing, but we have shown the most important part. □

We often assume $\mathcal{U}$ is differential, but this is not a consequence of rationality and continuity only.

We very often assume even more: that $\mathcal{U} : \mathcal{X} \to \mathbb{R}$ is $\mathcal{C}^2$ (twice differentiable, with continuous derivatives).

A classical counterexample is the *Leontiev* function: $\mathcal{U}(x_1, x_2) = \min(x_1, x_2)$.



This is used to represent *perfectly complementary* goods (such as x_1 being "left shoes" and x_2 being "right shoes").

0.6 Utility maximization problem

Given some *wealth* w and a *vector or prices* $\begin{pmatrix} p_1 \\ \vdots \\ p_L \end{pmatrix}$ *which are taken as given* (this is a fundamental restriction that will hold in our context of *competitive market* - and which is invalid in the case of monopolistic or quasi-monopolistic buyers/sellers), then the *budget set* is defined as

$$B_{p,w} = \left\{ x \in \mathbb{R}^L_+ : px \leq w \right\}.$$

The *utility maximization problem* is:

$$\max_x \mathcal{U}(x) \qquad \text{subject to } x \in B_{p,w},$$

and the solution to this problem is usually denoted as

$$X(p,w) \subset \mathbb{R}^L_+.$$

In line of principle, it could be empty. Next time, we will show that, given the assumptions made so far, it won't.

01/14/2011

Remark 2. *The utility representation is not unique: if $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $f \circ \mathcal{U}$ is equivalent to $\mathcal{U}$, since*

$$f(\mathcal{U}(x)) \geq f(\mathcal{U}(y)) \iff \mathcal{U}(x) \geq \mathcal{U}(y) \iff x \succeq y.$$

In other terms, utility has an *ordinal*, rather than *cardinal*, meaning.

An obvious consequence is that utilities can take indifferently negative, positive or mixed values.

It is interesting to notice that philosophically utility was first conceived as a sort of *measure of pleasure*. This approach has however being rejected - at least until recently, nowadays some attempts in neuroeconomics can be seen as possible renewed attempts in this direction.

Of course, in welfare economics we will *have to* make comparisons between people - comparing for instance marginal utility of some policy. This is something that economy however refrains from doing, as long as it is possible - it is more a matter of *politics*: for the moment, numbers just mean nothing to us, *relations do*.

Yesterday, we stated the UMP (Utility Maximization Problem), which can analogously written as:

$$\max_x \mathcal{U}(x) \qquad \text{subject to} \quad \begin{matrix} px \leq w \\ x_1 \geq 0 \\ \vdots \\ x_L \geq 0 \end{matrix}$$

which is a clear case in which we can apply the Kuhn Tucker theorem.

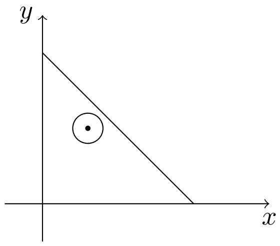
$$\frac{\partial \mathcal{U}(x^*)}{\partial x_l} = \lambda p_l + \lambda_l(-1)$$

with

$$\lambda = 0 \text{ if } px < w,$$

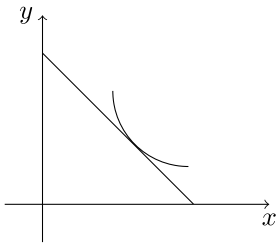
$$\lambda_l = 0 \text{ if } x_l > 0.$$

However, in our framework the first case just *won't* happen: it represents an internal solution, which would go against the hypothesis of *local non-satiation*:

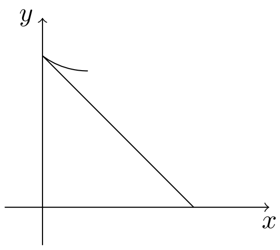


inside the circle we should be able to find some *preferred* bundle - and hence the point would not be a maximum. That's the reason why we will always assume that $\lambda \neq 0$.

We can instead have $\lambda_l = 0$. If it happens $\forall l$, then we say the solution is *interior* - though that strictly speaking is not true, since it will still be on the boundary given by the budget constraint:



If instead there's at least one l for which $\lambda_l > 0$, we will say we have a *corner solution*:



If we take the ratio of marginal utility for two different commodities, we get (in the optimum)

$$\frac{\frac{\partial \mathcal{U}(x)}{\partial x_l}}{\frac{\partial \mathcal{U}(x)}{\partial x_k}} = \frac{p_l}{p_k};$$

it can be also be thought as the opposite of the ratio of change:

$$-\frac{\partial x_k}{\partial x_l}$$

holding the utility constant. This can be seen as an application of the *implicit function theorem*.

The interpretation of the equation is straightforward: the optimum is the allocation such that the individual exchange rate corresponds to the *market* exchange rate. If instead we are not at the optimum, we can see it as *relative shadow price*.

By the way, the budget line:

$$p_1 x_1 + p_2 x_2 = w$$

can be expressed as

$$x_2 = \frac{w}{p_2} - \frac{p_1}{p_2} x_1;$$

hence, we have a corner solution with $x_2 = 0$ (and hence $x_1 = \frac{w}{p_1}$) - that is, an indifference curve that is *flatter* than the budget line, if

$$\frac{\frac{\partial \mathcal{U}(x)}{\partial x_1}}{\frac{\partial \mathcal{U}(x)}{\partial x_2}} < \frac{p_1}{p_2} \qquad \text{in} \left(\frac{w}{p_1}, 0\right).$$

We didn't mention one aspect: the fact that prices are the same for each agent is an important restriction, even when talking about relative prices: for instance, the number of bottles of beer that you can buy with 10 kilograms of rice is much higher in Corea than in Italy. In other words, we are forgetting the aspects of *transportation costs*, *takes* and so on.

However, inside a country we can probably claim that the restriction is not distorsive.

So finally, we can say that if $\mathcal{U}$ is continuous, and for $p >> 0$, then *a solution exists*! It is a consequence of the Weierstrass theorem, which holds since the consumption set is compact. We already called the solution $X(p, w)$ - however it can be a function or instead a *correspondence*.

If it is a function and it is *differentiable*, we can write

$$D_p X(p, w) = \begin{pmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_1}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L}{\partial p_1} & \cdots & \frac{\partial x_L}{\partial p_L} \end{pmatrix},$$

and likewise for the wealth

$$D_w X(p, w) = \begin{pmatrix} \frac{\partial x_1}{\partial w} \\ \vdots \\ \frac{\partial x_L}{\partial w} \end{pmatrix}.$$

We can now state the following result:

Proposition 1. *If* $\mathcal{U}$ *is continuous, locally non-satiated,* $p >> 0$ *and* $\mathcal{X} = \mathbb{R}^L_+$*, then* X *is* homogeneous of degree 0*, that is:*

$$X(\alpha p, \alpha w) = X(p, w);$$

intuitively, we can think it's like saying what matters is not if nominal wages change, but if real ones do*. This is sometimes called* lack of money illusion *- changing from Lira to Euro should not (have) change(d) anything in principle.*

Actually, local non-satiation is not really required for this result - it will instead be useful for following ones.

Theorem 3 (Walras' law)**.**

$$pX(p, w) = w,$$

or, intuitively, the consumer will always spend as much as he can (and that's where we use local non-satiation).

If $\succeq$ is convex, then $\mathcal{U}$ is quasi-concave. If this is the case, then $X(p, w)$ is a *convex set*: if $x', x'' \in X(p, w)$, then (without loss of generality)

$$px' \leq px'' \leq w$$

and hence for any linear combination

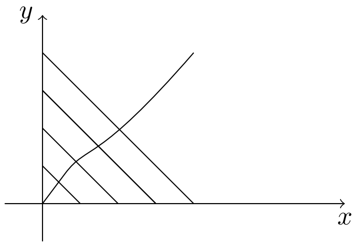
$$p(\alpha x' + (1 - \alpha)x'') \leq w;$$

on the other hand, being $\mathcal{U}$ quasi-concave, this linear combination is at least as good as x' and x'', and hence it must also be in $X(p, w)$.

Moreover, if we assume $\mathcal{U}$ is *strictly* quasi-concave, $X(p, w)$ can simply not contain more than one point - the solution will be unique.

0.7 Comparative statics

If we study $X(p,w)$ with changing w (keeping $p=\underline{p}$), we get what is called the *Engel* function:



through a *wealth expansion path*:

$$\{X(\overline{p},w) : w > 0\}\,.$$

If we have

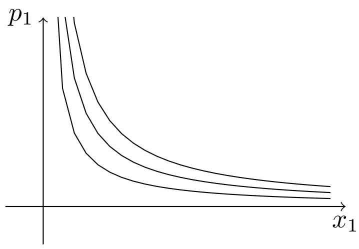
$$\frac{\partial X_l(p,w)}{\partial w} \geq 0,$$

then we say commodity l is normal (rather than *inferior*).

Some commodities can be *normal* in some cases and *inferior* in some others - for instance, McDonalds may be normal for lower wages, then become inferior at high wages. But when we say a good is simply "normal", we usually mean *for any wealth level*.

0.7.1 Demand curve

The demand curve is what happens to demand for commodity l as the price of the commodity changes:



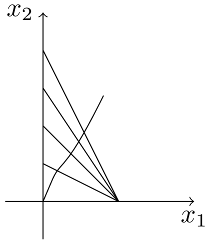
the typical curve is downward sloped:

$$\frac{\partial X_l(p,w)}{\partial p_l} < 0,$$

but there may be - theoretically, and also in practice - exceptions, such as *prestige* goods (but in those cases, it's hard to say if the good is really the same after price changes) or more generally *Giffen* goods: though the classical story (potatoes being a Giffen good in Ireland's history) is probably not correct, recent studies focused on the same phenomenon with rice in China. Giffen goods are *inferior* goods for which the *income effect* dominates the *substitution effect*.

0.7.2 Offer curve

The offer curve describes what happens to demand when the price of a single commodity changes:



0.8 Indirect utility function

This function is defined as

$$V(p,w) = \max_x \mathcal{U}(x) \qquad \text{subject to } \begin{matrix} px \leq w \\ -x \leq 0 \end{matrix}$$

or equivalently

$$V(p,w) = \mathcal{U}(X(p,w)).$$

Proposition 2. *Under usual assumptions on the utility (continuity, local non-satiation* [4]*,* $\mathcal{X} = \mathbb{R}^L_+$*), if we increase* w*, then* V *increases too (in the world we are studying, money gives happiness).*

Proposition 3. *Vice versa (but not entirely specularly),* V *is* not increasing *in* p_l *for any* l*. It may however remain constant (instead than decrease) if the quantity of the good bought at the original* p_l *is already 0.*

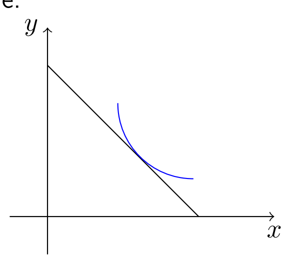
Proposition 4. *Still under the same assumptions,* V *is continuous and quasi-convex.*

1 The expenditure minimization problem

This is the *dual problem* of the utility maximization:

$$\min px \qquad \text{subject to } \begin{matrix} \mathcal{U}(x) \geq \mathcal{U}^* \\ x \in \mathcal{X}. \end{matrix}$$

Intuitively, when we are maximizing utility we are also minimizing our expenditure:



[4]More precisely, this is not required if we mean "*weakly* increases", while monotonicity is required if we mean "*strictly* increases".

It is interesting that this is not guaranteed instead if we have *thick* indifference curves - in that case the individual doesn't care about throwing away a given quantity of money/goods.

This problem also corresponds to some Kuhn-Tucker conditions:

$$\max -px \qquad \text{subject to} \begin{array}{c} -\mathcal{U}(x) \leq -\mathcal{U} \\ -x_l \leq 0 \; \forall l \end{array}$$

gives

$$-p_l = \lambda \left(-\frac{\partial \mathcal{U}}{\partial x_l} \right) + \lambda_l(-1).$$

Again, we will have $\lambda_l = 0$ if $x_l > 0$. Assuming $x_l > 0$ for each l, we have

$$p_l \geq \lambda \frac{\partial \mathcal{U}}{\partial x_l}.$$

Theorem 4. *Let's assume $p >> 0$. Then, we have that if x^* is a solution to the UMP with $w > 0$, then x^* is also a solution to the EMP with $\mathcal{U}^* = \mathcal{U}(x^*)$.*

Proof. Suppose x^* is a solution to UMP but not to EMP - let x' be a solution to EMP, which means $px' < px^*$ and

$$\mathcal{U}(x') \geq \mathcal{U}(x^*).$$

We can find a ball around x' which is entirely contained in the budget set. From local non-satiation, we know inside this ball there will be some x'' which gives strictly more utility than x'; so

$$\mathcal{U}(x'') > \mathcal{U}(x') \geq \mathcal{U}(x^*),$$

and hence we proved that x^* was not a solution to UMP: x'' is better and is feasible too.

Vice versa, suppose that x^* is a solution to EMP for $\mathcal{U}$ but not to UMP - let x' be a solution to UMP. Assume $\mathcal{U} > \mathcal{U}(0)$: then $x^* \geq 0$, and $x^* \neq 0$, $px^* > 0$.

Since x^* is not a solution to UMP, we must have x' such that

$$\mathcal{U}(x') > \mathcal{U}(x^*) = \mathcal{U}$$

and $px' \leq w = px^*$.

Let $x_\alpha = \alpha x'$ for $\alpha < 1$: for $\alpha \to 1$, we have that $\mathcal{U}(x_\alpha) > \mathcal{U}(x')$ by continuity. So we can find some $\hat{\alpha}$ for which

$$px_{\hat{\alpha}} < px' = w$$

and $\mathcal{U}(x_{\hat{\alpha}} \geq \mathcal{U}(x^*)$. This gives a contradiction, since $x_{\hat{\alpha}}$ is a better candidate for EMP than x^*.

□

$e(p, u)$ which solves the EMP is called the *expenditure function*.

The theorem we saw last time can be rewritten as:

$$e(p, v(p, w)) = w$$

and

$$v(p, e(p, u)) = u.$$

So, taking p as fixed, we can say that $v(p, \cdot)$ is the *inverse* of $e(p, \cdot)$.

Proposition 5. *Assume u continuous and locally non-satiated, and that $\mathcal{X} = \mathbb{R}^L_+$ (and recall that if prices aren't strictly positive, EMP may not have a solution).*

Then $e(\cdot, \cdot)$ is

1. *1 strictly increasing in u,*
2. *non-decreasing in p_l, $l = 1, \ldots, L$,*
3. *homogeneous of degree 1 in p,*
4. *concave,*
5. *continuous.*

Proof. can be easily proved by contradiction: given $u' > u$, suppose $e(p, u') \leq e(o, u)$, and let's assume that the related solutions are x' and x, that is, $u(x) = u)$ and $u(x') = u'$.

Then, by continuity, if we "scale" x by some $\alpha < 1$, we can get $u' > u(\alpha x') > u$.

So we have

$$p\alpha x' < px' = e(p, u') \leq e(p, u)$$

and this implies that x could not really be the solution to the minimization problem.

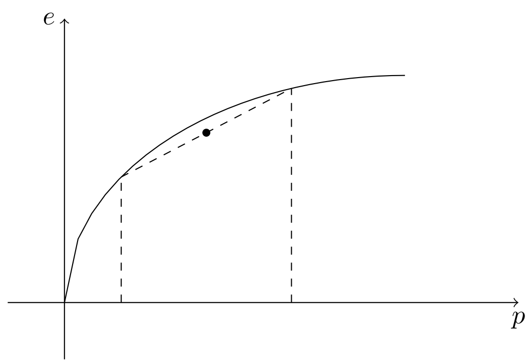
The proof of weak monotonicity in p_l is similar: let's assume $p'_l > p_l$, and let x' and x be the related solutions. Then,

$$e(p'u) = p'x' \geq px' \geq px = e(p, u).$$

Concavity in p means

$$e(\alpha p + (1 - \alpha)p', u) \geq \alpha e(p, u) + (1 - \alpha)e(p', u)$$

for $\alpha \in [0, 1]$.



Indeed, let x be the solution for p and x' for p': then

$$px'' \geq px$$
$$p'x'' \geq p'x'$$

and by multiplying respectively by α and $(1-\alpha)$, we get

$$\alpha px'' \geq \alpha px$$
$$(1-\alpha)p'x'' \geq (1-\alpha)p'x'$$

and hence

$$\alpha px'' + (1-\alpha)p'x'' \geq \alpha px + (1-\alpha)p'x'$$

which is precisely what we wanted to prove.

□

The solution (consumption bundle) to the EMP is $h(p,u)$: *Hicksian* or *compensated* demand. By the way, $h(p,u) \subset \mathbb{R}^L$: as in the case of the UMP, solutions can be multiple. This won't happen as long as we assume *strict convexity* of the utility function.

Proposition 6. *Let's assume u is continuous, non-satiated and $X = \mathbb{R}^L_+$, then*

- $h(\cdot,\cdot)$ *is homogeneous of degree* 0 *in* p,
- $u(h(p,\overline{u})) = \overline{u}$,
- *if u is quasiconcave, $h(p,\overline{u})$ is always a* convex *set,*
- *if u is strictly quasi-concave, $h(p,\overline{u})$ is always a singleton,*
- $x(p,e(p,u)) \equiv h(p,u)$,
- $h(p,v(p,w)) \equiv x(p,w)$.

The last identity kind of explains why we call h the *compensated* demand: it's what we ought to give to an individual if we want him to attain a given utility level.

Theorem 5 (Compensated law of demand)**.** *The following equality holds for all prices p'', p':*

$$[p'' - p']\,[h(p'',u) - h(p',u)] \leq 0.$$

Proof.

$$p''h(p'',u) - p''h(p',u) \leq 0$$

since the second term "optimizes h for the wrong prices".

Now let's assume p'' and p' differ in the l-th component: then

$$[p''_l - p'_l]\,[h_l(p'',u) - h_l(p',u)]\,;$$

in words, *"if the price of some commodity increases, the compensated demand for it cannot increase"* (keeping other prices fixed). There is no "Giffen goods" case for the compensated demand. □

Proposition 7. *Let $\overline{u}$ be continuous, locally non-satiated, strictly quasi-concave and $\mathcal{X} = \mathbb{R}^L_+$.*

Then,

$$h(p,u) = \nabla_p e(p,u) = \begin{pmatrix} \frac{\partial e(p,u)}{\partial p_1} \\ \vdots \\ \frac{\partial e(p,u)}{\partial p_L} \end{pmatrix}.$$

Proof. For an easier proof, we assume h is differentiable and $h(p,u) >> 0$.

We use the identity

$$e(p,u) \equiv ph(p,u)$$

and differentiate both sides:

$$\nabla_p e(p,u) = h(p,u) + \left(p^T, D_p h(p,u)\right)^T.$$

Now, we know that for an interior solution (and we did assume $h(p,u) >> 0$),

$$p = \lambda \nabla u(x^*) = \lambda \nabla u(h(p,u)) = \lambda \left(\nabla u(h(p,u)) \cdot D_p h(p,u)\right)^T.$$

So recalling that $u(h(p,\overline{u})) = \overline{u}$, we get that differentiating in p,

$$\nabla u(h(p,u)) \cdot D_p h(p,u) = 0;$$

the end of the proof is then an application of the envelope theorem.

□

This result has some interesting consequences:

- if h is differentiable, then

  $$D_p h(p,u) = D^2_p e(p,u) \in \mathbb{R}^{L \times L}$$

  and since e is concave, this is a negative semi-definite matrix. Those are often used properties by microeconometricians (another strong assumption is the existence of a *representative consumer*). This is also a *symmetric* matrix, and that implies:

  $$\frac{\partial h_e(p,u)}{\partial p_l} = \frac{\partial h_l(p,u)}{\partial p_e}.$$

1.0.1 Net substitutes and net complements

Two commodities are *substitutes* if the consumption of one increases when the price of the other does, and *complements* if the converse happens. We talk about *gross* substitutes/complements if we are considering the demand, and *net* substitutes/complements if we are considering *compensated* demand.

01/21/2011

1.1 Slutsky equation

Theorem 6.

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial X_l(p,w)}{\partial p_k} + \frac{\partial X_l(p,w)}{\partial w} X_k(p,w)$$

(at $u = v(p,w)$, or equivalently $w = e(p,u)$).

Proof.

$$h_l(p,u) = X_l(p, e(p,u));$$

if we differentiate in p, we get

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial X_l(p,e(p,u))}{\partial p_k} + \frac{\partial X_l(p,e(p,u))}{\partial w} \frac{\partial e(p,u)}{\partial p_k}$$

but we know (from yesterday) that

$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u)$$

and combining with the equation above we get

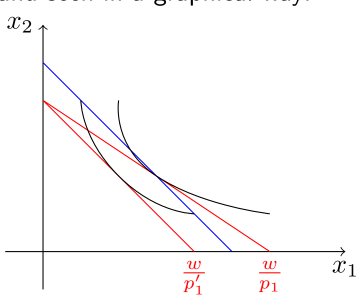
$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial X_l(p,w)}{\partial p_k} + \frac{\partial X_l(p,w)}{\partial w} X_k(p,w)$$

and that is the Slutsky equation. □

The Slutsky equation is often rewritten in another form:

$$\frac{\partial X_l(p,w)}{\partial p_k} = \frac{\partial h_l(p.u)}{\partial p_k} - \frac{\partial X_l(w)}{\partial w} X_k(p,w)$$

and seen in a graphical way:



(because of the law of composite demand, when a price raises the consumer will consume less of that good).

We are saying the consumer "we change the price of commodity 2 but we give you enough money so that your utility remains unchanged".

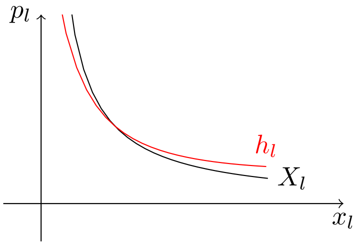
Now we can also see why there is the (at least theoretical) possibility of Giffen goods: if the last component in the Slutsky equation (without the minus sign) is "very negative", then the left hand side can be positive. This shows that a Giffen good is *necessarily* an inferior good (one for which that last component - change in consumption caused by change in wage - is negative).

So to resume:

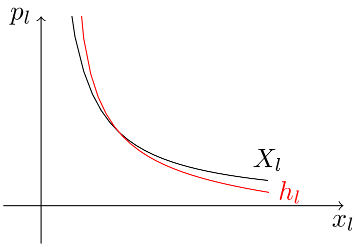
$$\text{Normal good} \qquad \frac{\partial X_l}{\partial w} \geq 0 \implies \frac{\partial X_l}{\partial p_l} < 0$$

$$\text{Inferior good} \qquad \frac{\partial X_l}{\partial w} < 0 \implies \begin{cases} \frac{\partial X_l}{\partial p_l} > 0 & \text{Giffen good} \\ \frac{\partial X_l}{\partial p_l} \leq 0 \end{cases}$$

Normal good:



Inferior good:



1.2 Uncertainty

We will start with the Von Neumann-Morgenstern theory: the basic consumption good are lotteries.

A lottery is a function with values in an *outcome space*. It is described by two elements: the outcome space itself and a *probability distribution* on it.

We will assume, for the moment, that the outcome set is finite:

$$\{c_1, c_2, \dots, c_N\}$$

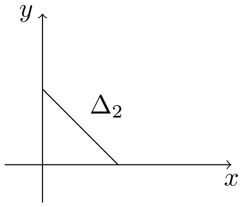
(where we usually assume those are amounts of money), so that a lottery is defined simply by a vector of probabilities:

$$\{p_1, p_2, \dots, p_N\} \text{ such that} \sum_{i=1}^{N} p_i = 1.$$

In other words, we can say a lottery is

$$P \in \underbrace{\Delta_N}_{\substack{n-1 \\ \text{dimensional} \\ \text{symplex}}} = \left\{ P \in \mathbb{R}_+^N : \sum_{n=1}^{N} p_n = 1 \right\}.$$

For instance,



We will call $\mathcal{L}$ the set of all lotteries, and assume the existence of a rational preference relation $\succeq$ on $\mathcal{L}$.

One may think that lotteries are not very important... but economists think instead that *everything* is a lottery: every action with uncertain consequences.

Different individuals often disagree on the probability values of a given lottery, but we will not consider this point for the moment, and from now on we assume they agree on them.

1.2.1 Compound lotteries

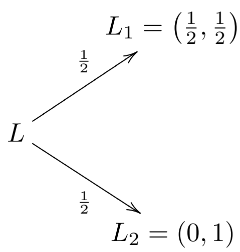
We could imagine a lottery which outcomes are the possibility to play other lotteries. . . and more in general

$$(L_1, \ldots, L_k, \Pi_i, \ldots, \Pi_K)\,;$$

then, if for instance we assume $\left(L_k = \left(p_1^k, \ldots, p_N^k\right)\right)$, we can consider a *reduced* lottery:

$$L = \sum_{k=1}^{K} \Pi_k L_k;$$

for instance, in the lottery



the probability of getting 2 is

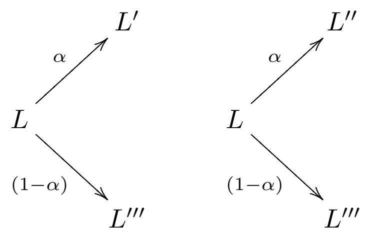
$$\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}.$$

We will make two main assumptions on lotteries:

1. *continuity*:
$$\forall L', L'', L'''$$
$$\{\alpha \in [0,1] : \alpha L' + (1-\alpha)L'' \geq L'''\}$$
and
$$\{\alpha \in [0,1] : L''' \geq \alpha L' + (1-\alpha)L''\}$$
are closed.

2. *independence axiom*:

$$\forall L', L'', L''' \qquad L' \succeq L'' \iff \alpha L' + (1-\alpha)L'' \succeq \alpha L'' + (1-\alpha)L'''$$



This axiom has several consequence, the most immediate being:
$$L' \sim L'' \iff \alpha L' + (1-\alpha)L''' \sim \alpha L'' + (1-\alpha)L'''$$
($\sim$ can be seen as just $\overset{\succeq}{\preceq}$) and
$$L' \succ L'' \iff \alpha L' + (1-\alpha)L''' \succ \alpha L'' + (1-\alpha)L'''$$
($\succ$ can be seen as $\succeq$ but $\not\preceq$).

An utility function U has the *Neumann-Morgestern expected utility form* if there are numbers $u_1, \ldots, u_N$ such that
$$U(L) = \sum_{n=1}^{N} p_n u_n \qquad \forall L \in \mathcal{L}.$$
We can see this form as an *expected utility*, and this is the way most models will design expectancies.

Obviously, not all preferences will have this form: for instance, if
$$\widetilde{U}(L) = [U(L)]^3$$
then $\widetilde{U}$ and U represent the same preferences, but $\widetilde{U}$ cannot be written in the form described.

Theorem 7. *U is a Von Neumann-Morgenstern expected utility form representation of $\succeq$ if and only if U is linear.*

Proof. $\implies$:

$$\begin{aligned} Y(\beta L+(1-\beta)L) =& \sum_{n=1}^{N} u_n[\beta(p_n+(1-\beta)p_n] \\ =& \beta\sum_{n=1}^{N} u_n p_n + (1-\beta)\sum_{n=1}^{N} p'_n u_n \end{aligned}$$

Vice versa, if U is linear, let's consider

$$L_n := (0,\dots,0,\underbrace{1}_{n},0,\dots,0)$$

(also called *degenerate* lotteries). They are *extreme* lotteries, in the sense that they are the extrema of the convex set of all lotteries, which can be obtained as linear combinations of them:

$$L = \sum_{n=1}^{N} p_n L_n$$

now,

$$U(L) = U\left(\sum_{n=1}^{N} p_n L_n\right)$$

and linearity tells us

$$U(L) = \sum_{n=1}^{N} p_n U(L_n) = \sum_{n=1}^{N} p_n u_n$$

which is precisely the Von Neumann-Morgenstern form.

□

Not all transformations preserve the expected utility format of a given utility function:

Proposition 8. *suppose U is a Von Neumann-Morgenstern expected utility representation of $\succeq$, then $\widetilde{U} = F(U)$ is another one if and only if*

$$\exists\beta > 0, \gamma : \qquad \widetilde{U}(L) = \beta U(L) + \gamma$$

For instance

$$F(x) := \beta x + \gamma$$

is precisely an *affine* transformation - that is, satisfies the condition imposed - while as seen above

$$F(x) := x^3$$

doesn't.

Proof. If $\widetilde{U}(L) = \beta U(L) + \gamma$ then

$$\widetilde{U}(L) = \beta \sum_{n=1}^{N} p_n u_n + \gamma = \sum_{n=1}^{N} \underbrace{(\beta u_n + \gamma)}_{\widetilde{u}_n} p_n.$$

Vice versa, suppose that $\widetilde{U}$ is another V-M expected utility representation: then, we can find $\overline{L}$ and $\underline{L}$ such that $\forall L \in \mathcal{L}$ we have $\overline{L} \succeq L \succeq \underline{L}$ (those values exist because we are analyzing a continuous function over a compact set).

If $\overline{L} \sim \underline{L}$, the utility function is constant and the proof is obvious.

Let's hence assume that $\overline{L} \succ \underline{L}$, and define, for any lottery $L \in \mathcal{L}$,

$$\lambda_L : \qquad \lambda_L U\left(\overline{L}\right) + (1 - \lambda_L) U(\underline{L}) = U(L)$$

or analogously:

$$\lambda_L := \frac{U(L) - U(\underline{L})}{U(\overline{L}) - U(\underline{L})}.$$

Now, since the two utilities represent the same preferences and we have defined λ_L such as

$$L \sim \lambda_L \overline{L} + (1 - \lambda_L)\underline{L},$$

we have

$$\widetilde{U}(L) = \widetilde{U}\left(\lambda_L \overline{L} + (1 - \lambda_L)\underline{L}\right)$$

and since $\widetilde{U}$ is linear, the above is equal to

$$\begin{aligned}
\lambda_L \widetilde{U}(\overline{L}) + (1 - \lambda_L)\widetilde{U}(\underline{L}) =& \lambda_L \left[\widetilde{U}(\overline{L}) - \widetilde{U}\left(\underline{L}\right)\right] + \widetilde{U}(\underline{L}) \\
=& \frac{U(L) - U(\underline{L})}{U(\overline{L}) - U(\underline{L})} \left[\widetilde{U}\left(\overline{L}\right) - \widetilde{U}(\underline{L})\right] + \widetilde{U}(\underline{L}) \\
=& \underbrace{\frac{\widetilde{U}\left(\overline{L}\right) - \widetilde{U}(\underline{L})}{U(\overline{L}) - U(\underline{L})}}_{\beta} U_L + \underbrace{\widetilde{U}(\underline{L}) - \frac{\widetilde{U}(L) - \widetilde{U}(\underline{L})}{U(\overline{L}) - U(\underline{L})} \widetilde{U}(\underline{L})}_{\gamma}
\end{aligned}$$

□

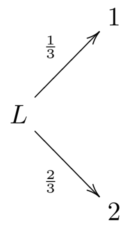
1.3 Distribution functions

Let's take the general case in which C, the outcomes set, is $\mathbb{R}_+$. That means that a lottery will now be a distribution function on $\mathbb{R}_+$, that is, a function

$$F : \mathbb{R} \to [0, 1]$$

where we interpret $F(x)$ as the probability that we get an outcome $c \leq x$.

For example (back in the discrete case), given



we have

$$F\left(\frac{1}{2}\right) = 0, \qquad F(1) = \frac{1}{3}, \qquad F(2) = 1.$$

In place of the Von Neumann-Morgenstern expected utility representation, we will now analogously assume:

$$U(F) = \int u(x) dF(x).$$

In general, if F has a *density function* f, we can write

$$F(x) = \int_{-\infty}^{x} f(y) dy$$

and hence

$$U(F) = \int u(x) f(x) dx:$$

we call $u(x)$ the *Bernouilli utility function*, whatever shape it takes.

Risk aversion: given a generic Bernouilli function, we say that if

$$U\left(\int x dF(x)\right) \geq \int u(x) dF(x)$$

(or analogously U is concave), then the individual is risk averse.

01/27/2011

Theorem 8 (Expected utility theorem)**.** *Given $\mathcal{L}$ the set of all lotteries, $L = (p_1, \ldots, p_n)$, and $\succeq$ on $\mathcal{L}$, rational, continuous and that satisfies the independence axiom. Then, $\succeq$ has a V.N-M Expected Utility Representation: a mapping $U : \mathcal{L} \to \mathbb{R}$ such that*

$$U(L) \geq U(L') \iff L \succeq L'.$$

Proof. 1. If $L' \succ L$ and $\alpha \in (0, 1)$, then

$$L' \succ \alpha L' + (1 - \alpha) L \succ L.$$

Indeed, this is true because

$$L' = \alpha L' + (1 - \alpha) L' \succ \alpha L' + (1 - \alpha) L \succ \alpha L + (1 - \alpha) L = L$$

(where the inequalities are applications of the independence axiom - which, we had seen, holds with both "$\succ$" and "$\succeq$").

2. Then, we had seen that we have $\overline{L}$ and $\underline{L}$ such that

$$\overline{L} \succeq L \succeq \underline{L} \qquad \forall L \in \mathcal{L};$$

if $\overline{L}$ and $\underline{L}$ are the same, then the proof is trivial:

$$U(L) = a = \sum p_n a_n.$$

Let's assume instead $\overline{L} \succ \underline{L}$, then

$$\beta\overline{L} + (1-\beta)\underline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L} \iff \beta > \alpha.$$

Indeed, assuming $\beta > \alpha$,

$$\beta\overline{L} + (1-\beta)\underline{L} = \gamma\overline{L} + (1-\gamma)\left[\alpha\overline{L} + (1-\alpha)\underline{L}\right]$$

implies

$$\gamma + (1-\gamma)\alpha = \beta \implies \gamma = \frac{\beta - \alpha}{1-\alpha} > 0.$$

Now, since we knew

$$\overline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L},$$

then

$$\gamma\overline{L} + (1-\gamma)\left[\alpha\overline{L} + (1-\alpha)\underline{L}\right] \succ \alpha\overline{L} + (1-\alpha)\underline{L}.$$

Vice versa, if indeed

$$\beta\overline{L} + (1-\beta)\underline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L}$$

then we cannot have $\beta = \alpha$ (otherwise the two lotteries would be the same). $\beta < \alpha$ is also impossible, because we have just shown that it would imply the opposite relation between composite lotteries. So $\beta > \alpha$.

3. $$\forall L \; \exists! \alpha_L \; : \alpha_L\overline{L} + (1-\alpha_L)\underline{L} \sim L$$

We do know that

$$\underline{A} := \left\{\alpha \in [0,1] | L \succeq \alpha\overline{L} + (1-\alpha)\underline{L}\right\}$$

is non-empty (it contains at least 0) and closed (because $\succeq$ is closed by assumption). The same holds for

$$\overline{A} := \left\{\alpha \in [0,1] | L \preceq \alpha\overline{L} + (1-\alpha)\underline{L}\right\}$$

(which contains at least 1). But $\underline{A} \cup \overline{A} = [0,1]$, which is compact, so their intersection cannot be empty: there must be some point α_L which is in both - that is, for which the equality

$$L \sim \alpha_L\overline{L} + (1-\alpha_L)\underline{L}$$

holds (it contains *only* α_L, because if it contained another one - which we would assume with no loss of generality being greater - then applying what we showed in the former step we would have that equality could *not* hold for it).

4. $u(L) := \alpha_L$ represents the preference relation. This is a direct consequence of step 2:

$$\begin{aligned}\alpha_L \geq \alpha_{L'} &\iff \alpha_L(\overline{L}) + (1-\alpha_L)(\underline{L}) \succeq \alpha_{L'} L(\overline{L}) + (1-\alpha_{L'})(\underline{L}) \\ &\iff L \succeq L'.\end{aligned}$$

5. $u(L) = \alpha_L$ is *linear*. It is sufficient to show that

$$u(\beta L + (1-\beta)L') = \beta u(L) + (1-\beta)u(L') \;\forall \beta \in [0,1].$$

Now,

$$\beta L + (1-\beta)L' \sim \beta L + (1-\beta)\left[u(L')\overline{L} + (1-u(L')\underline{L})\right]$$

as consequence of independence axiom, and using this axiom again we get

$$\beta L + (1-\beta)\left[u(L')\overline{L} + (1-u(L')\underline{L})\right]$$

$$\begin{aligned}&\sim \beta\left[u(L)\overline{L} + (1-u(L))\underline{L}\right] + (1-\beta)\left[u(L')\overline{L} + (1-u(L'))\underline{L}\right] \\ &= \left[\beta u(L) + (1-\beta)u(L')\right]\overline{L} + \left[\beta(1-u(L)) + (1-\beta)(1-u(L'))\right]\underline{L}\end{aligned}$$

but then, the utility of this last utility is precisely

$$\beta u(L) + (1-\beta)u(L')$$

which is what we wanted to show.

□

As seen last time, we will often assume that a lottery doesn't have a finite set of possible outcomes, but instead $C = \mathbb{R}_+$. A lottery L will hence be a distribution function on $\mathbb{R}+$ and we will take

$$\mathbb{P}(X \leq x) = F(x)$$

with

$$U(F) = \int u(x)dF(x)$$

being the *expected utility* (and u being the *Bernoulli utility function*, usually increasing).

We had seen the definition of *risk aversion*: given a degenerate lottery $F_E(x)$ which gives as outcome

$$\int x dF(x)$$

with probability 1, a risk averse individual will always (weakly) prefer F_E to F:

$$F_E \succeq F;$$

in particular, if $F_E \sim F$ (again, for all $F \in \mathcal{L}$), we say the individual is *risk-neutral*.

We say an individual is *strictly* risk-averse if

$$\forall F \text{ such that } F \neq F_E, \qquad F_E \succ F$$

and finally that he is risk-lover if

$$\forall F \; F_E \preceq F$$

(and we define *strict* risk love analogously as in the risk aversion case).

By the way,

$$F_{EE} = F_E.$$

It is obvious that the expected utility of a degenerate lottery is just the utility of its (sure) outcome: in particular, given F, this translates to

$$\int u(x) dF_E(x) = U\left(\left[\int x dF(x)\right] \cdot 1\right);$$

this directly implies that the definition given of risk aversion is the same we had seen last time.

Categorizing individuals in terms of risk aversion is interesting. For instance, we could say that people have to be risk lover to take this course.

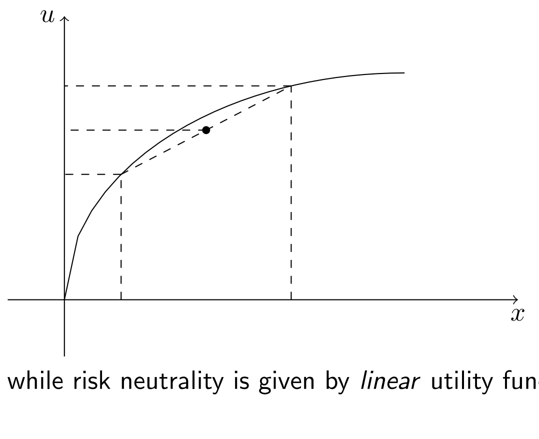
When people have more wealth, we see they exhibit less risk aversion: we are observing *decreasing risk aversion*. But risk aversion is usually still strictly positive for anyone. This explains the existence of the many existing types of *insurance*.

The classical theory doesn't consider (and hence we won't, neither) *asymmetric* behaviours such as *loss* aversion.

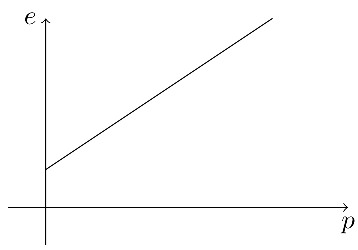
It can be shown that

$$u\left(\int x F(x)\right) \leq \int u(x) dF(x) \qquad \forall F$$

if and only if u is *concave*:



while risk neutrality is given by *linear* utility functions:



and risk loving utilities are *concave*.

There are other ways to define risk aversion:

Definition 9 (Certainty equivalence)**.** $C(F,u)$ *is defined implicitly by*

$$u(c(F,u)) = \int u(x)dF(x).$$

If one is (strictly) risk averse, we have that

$$c(F,u) \leq \int xdF(x).$$

This is an interesting concept, since it gives us a measure of the *risk prize*: how much an individual will be willing to pay in order to neutralize risk.

Finally, we can define the:

Definition 10 (probability premium)**.**

$$\Pi(x,\varepsilon,u): \qquad u(x) = \left[\frac{1}{2} + \Pi(x,\varepsilon,u)\right] u(x+\varepsilon) + \left[\frac{1}{2} - \Pi(x,\varepsilon,u)\right] u(x-\varepsilon):$$

it measures how much the probabilities should change *in order to make the lottery attractive for the individual. If* $\Pi(x,\varepsilon,u) > 0\ \forall x,\ \forall \varepsilon > 0$*, then the individual is risk averse.*

1.4 Measures of risk aversion

Definition 11 (Coefficient of absolute risk aversion)**.**

$$r_A(x,u) = -\frac{u''(x)}{u'(x)}:$$

given that $u'(x) > 0$*, we easily see that "the more"* u *is concave (*$u''(x) < 0$*), the higher is* r_A*.*

There's a particular utility function:

$$u(x) = -e^{-\alpha x}$$

which has a *constant* coefficient of absolute risk aversion:

$$u'(x) = \alpha e^{-\alpha x} \implies u''(x) = -\alpha^2 e^{-\alpha x} \implies r_A = \alpha:$$

this is probably not the utility function of Bill Gates... but it's frequently used in finance.

01/28/2011

We can talk about people being more or less risk averse, and make comparison among individuals.

Given u_1, u_2, the following notions are equivalent:

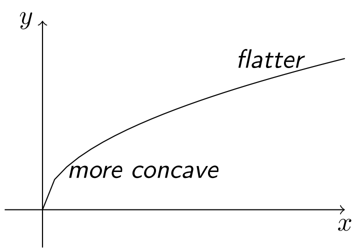
1. $r_A(x, u_2) \geq r_A(x, u_1)\ \forall x$
2. $c(x, u_1) \geq c(x, u_2)\ \forall x$
(a more risk averse individual will value more a given lottery rather than a certain outcome, if compared to a less risk averse one)
3. $\Pi(x, \varepsilon, u_1) \leq \Pi(x, \varepsilon, u_2)\ \forall x,\ \forall \varepsilon \in (0, x)$
(the condition on x guarantees that $x - \varepsilon \not< 0$)
4. $\int u_2(x)\, dF(x) \geq u_2(\overline{x}) \implies \int u_1(x)\, dF(x) \geq u_1(\overline{x})$
(if I do accept a lottery rather than a given fixed sum, then some guy more risk-lover than me make the same choice)
5. u_2 is a increasing concave transformation of u_1 (there exists some $f : \mathbb{R} \to \mathbb{R}$ increasing and concave such that $u_2(x) = f(u_1(x))$).
For instance, if we assume $f(x) = \sqrt{x}$, then given $u_1(x) = ax$ (linear and hence risk neutral), $u_2 = f(u_1(x)) = \sqrt{ax}$ is (more) risk averse (than u_1).

By the way, for a risk neutral individual r_A is zero, $c(x, u)$ is simply the expected value of the lottery, while Π is again zero.

1.5 Decreasing risk aversion

Definition 12. *u is exhibiting* decreasing absolute risk aversion *if the coefficient of risk aversion is decreasing in x:*

$$\frac{\partial r_A}{\partial x} \leq 0$$



Proposition 9. *The following are equivalent:*

1. *decreasing absolute risk aversion,*

2. *if we define* u_1, u_2 *as*

$$u_1(z) := u(x_1 + z)$$
$$u_2(z) := u(x_2 + z)$$

with

$$x_1 > x_2$$

then u_2 *is an increasing and concave transformation of* u_1 *(in other terms, if I start from a higher wealth I'm less risk averse),*

3. $$\int u(x_2 + z)\, dF(z) \geq u(x_2) \implies \int u(x_1 + z)\, dF(z) \geq u(x_1) \qquad \forall F,$$

4. *defined* c_x *as*

$$u(c_x) = \int u(x + z)\, dF(z)$$

then $x - c_x$ *is decreasing in* x*. Indeed,* $x - c_x$ *can be seen as* how much I'm willing to pay to avoid participating in the lottery.

We talked about *absolute* risk aversion. . . there is also the concept of *relative* risk aversion.

Let's take a distribution $F(t)$ and consider the lottery which gives me $x \cdot t$, where I bet x, all my current fortune.

Now, given

$$\widetilde{u}(t) := u(tx)$$

we have that

$$\widetilde{u}'(t) = xu'(tx)$$

and

$$\widetilde{u}''(t) = x^2 u''(tx).$$

So

$$r_A(1, \widetilde{u}) = -\frac{x^2 u''(x)}{xu'(x)} = -\frac{xu''(x)}{u'(x)}$$

and this is the coefficient of *relative risk aversion* for u.

As happens with the absolute one, there is a precise functional form which gives a *constant* relative risk aversion:

$$u(x) = Ax^{1-\alpha} : \qquad \alpha \in (0, 1).$$

Indeed,

$$u(x) = Ax^{1-\alpha}$$
$$\Downarrow$$
$$u'(x) = (1 - \alpha)Ax^{-\alpha}$$

$$u''(x) = -\alpha(1-\alpha)Ax^{-\alpha-1}$$

and hence

$$r_R(x,u) = \alpha$$

Remark 13. *If $r_R(x,u)$ is decreasing in x, the same can by said about $r_A(x,u)$.*

Proposition 10. *The following are equivalent:*

1. *$r_R(x,u)$ is decreasing in x,*
2. *if $u_1(t) = u(tx_1), u_2(t) = u(tx_2)$ and $x_1 \geq x_2$ then u_2 is an increasing concave transformation of u_1,*
3. *if we define*
$$u(\widetilde{c}_x) = \int u(x \cdot t)\ dF(t)$$
with F probability distribution on $[0,\infty]$, then
$$\frac{x}{c_x}$$
is decreasing in x,
4. *(still with $x_1 \geq x_2$)*
$$\int u(x_2 t)\ dF(t) \geq u(x_2) \implies \int u(x_1 t)\ dF(t) \geq u(x_1)$$

Example 14. *Let's consider the following situation: we can choose to invest α in a risky lottery, and retain β (with $\alpha + \beta = w$).*

Our problem will be

$$\max_{0 \leq \alpha \leq w} \int u(w + \alpha(z-1))\ dF(z).$$

For interior solutions ($\alpha \in (0,w)$) we must have

$$\int u'\left[w + \alpha(z-1)\right](z-1)\ dF(z) = 0$$

for $\alpha = 0$ to be a solution, instead, we must have

$$\int u'[w + \alpha^*(z-1)](z-1)\ dF(z) \leq 0$$

and for $\alpha = w$ to be a solution,

$$\int u'[w + \alpha^*(z-1)](z-1)\ dF(z) \geq 0.$$

If we assume constant absolute risk aversion ($u(x) = -e^{-\gamma x}$), this translates to

$$
\begin{aligned}
\int \gamma e^{-\gamma[w+\alpha(z-1)]}(z-1)\,dF(z) =& \int \gamma e^{-\gamma w} e^{-\gamma\alpha(z-1)}(z-1)\,dF(z) \\
=& \gamma e^{-\gamma w} \int e^{-\gamma\alpha(z-1)}(z-1)\,dF(z) \\
=& 0
\end{aligned}
$$

so we will choose the optimal α independently from the actual level of w: how much we invest in a risky asset does not *depend on our wealth.*

On the other hand, if we have decreasing *coefficient of absolute risk aversion, α will* decrease *in w.*

In reality, people may very well *disagree* on the probability of different events, and there is a theory about individual expectations.

In the discrete case, we have something very similar to what we had before:

$$u_i(L) = \sum_{n=1}^{N} u_n^i p_n :$$

individuals $i = 1$ and $i = 2$ will hence possibly have different *Bernoulli functions*. But the individuals may disagree also on the *probabilities* of some outcome:

$$u_i(L) = \sum_{n=1}^{N} u_n^i p_n^i .$$

We often talk about *rational expectations* if $p_n^i \equiv p_n$: this can be seen as the absence of *private information*.

Often, a distinction is made between *risk* and *uncertainty*: the former denoting situations in which probabilities are known, the latter situations in which they are not.

Example 15. *Let's consider two urns:*

1. *containing 49 white balls and 51 black ones,*
2. *containing 100 balls black or white .*

Let's assume an individual is asked to extract a ball from one urn, and he will get $100\$$ if the ball is black. What urn will he chose? This is not just a matter of risk aversion, but of how uncertainty *is faced.*

The Ellsberg paradox is the fact that individuals will choose the first urn rather than the second both if their aim is to extract a black ball, *and* if it is instead to (in the sense that they are paid if they) extract a white ball, *without changing the urn between the two choices*. So the expected utility model is not sufficient to model the way people choose under uncertainty: people in this thought experiment are *not* just guessing some probability distribution and then acting consequently.

2 Production

Just as we have seen for the consumer, we can assume firms have too a choice set $Y \subseteq \mathbb{R}^L$ ("production set") and preference relations on it.

If we have a feasible choice

$$y = (y_1, y_2, \dots, y_L) \in Y$$

then $Y_l > 0$ is an output, $Y_l < 0$ is an input (and $Y_l = 0$ simply means the type of good doesn't take part of the production process).

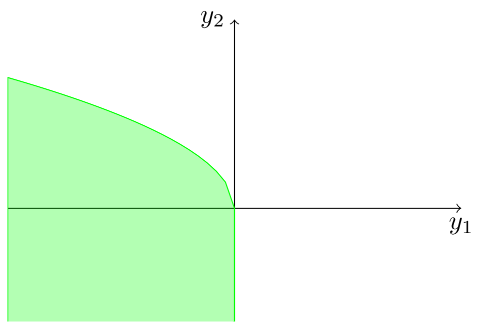
Given some amount of some output, there may be different combinations of inputs that allow to create it: for instance to produce a table I can use some hours of carpenters' work or of machine work, and in the second case the energy used to power the machine will be an additional input...

We have a *transformation function* F to describe Y:

$$F(y) \leq 0 \iff y \in Y$$

and

$$F(y) = 0 \iff u \in \delta Y.$$



The picture represents a typical feature: if carpenters' work hours can produce tables, that doesn't mean that from tables I can produce carpenters' work hours.

The firm can choose any y from Y and it will choose according to *gain maximization*: given prices $p = (p_1, p_2, \dots, p_L)$, it will solve

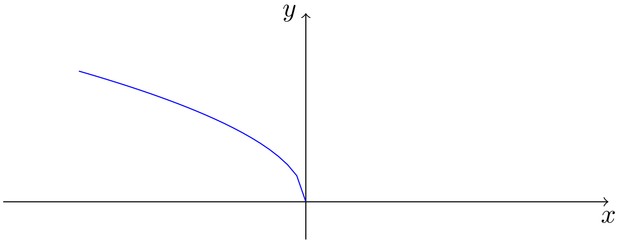
$$\max_y py \qquad \text{subject to } y \in Y.$$

A particular example of f which is very often used is the one of a *single output*:

$$q = f(z_1, z_2, \dots, z_{L-1})$$

in which we sometimes (but it depends!) also assume inputs are denoted in *absolute value* (so they are positive quantities). In our book, inputs are usually denoted by negative values for simplicity of calculations, so an example of production function can be

$$q = \sqrt{-z}$$



where z is implicitly assumed to be in $\mathbb{R}_-$.

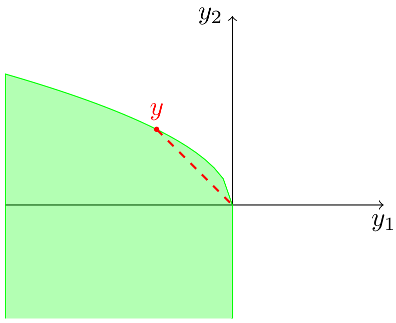
2.1 Frequent assumptions about Y

1. $Y \neq \emptyset$,
2. Y is closed,
3. "no free lunch": $Y \cap \mathbb{R}^L_+ \subset \{0\}$ (we can not produce some positive quantity of something without *consuming* some quantity of something else),
4. *free disposal*: $Y - \mathbb{R}^L_+ = Y$ (if I can produce, I can also produce something less), or in other terms:

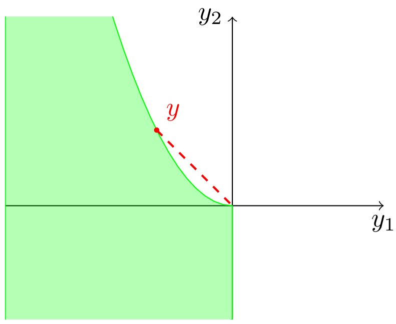
$$y \in Y, y' \leq y \implies y' \in Y$$

5. *irreversibility*: $Y \cap -Y \subset \{0\}$ (what we saw above about tables and carpenters' work hours),
6. *non increasing returns to scale* (a frequent, thought sometimes not made, assumption):

$$y \in Y \ \wedge \ \alpha \in [0,1] \implies \alpha y \in Y.$$



the utility represented above fulfills this assumption, while the one below doesn't:



7. the above implies *possibility of inaction*: $0 \in Y$. This is not a trivial assumption in reality, since for instance there can be commitments to both inputs and outputs,

8. *non-decreasing* returns to scale:

$$y \in Y \ \wedge \ \alpha \geq 1 \implies \alpha y \in Y$$

(this assumption doesn't hold in the picture for which non-increasing returns to scale held, and vice versa),

9. *constant returns to scale*:

$$\forall y \in Y \ \forall \alpha \geq 0 \qquad \alpha y \in Y$$

(that means both *non-increasing* and *non-decreasing* returns to scale - the border of Y is linear),

10. *additivity*:

$$Y + Y \subset Y,$$

11. *convexity*:

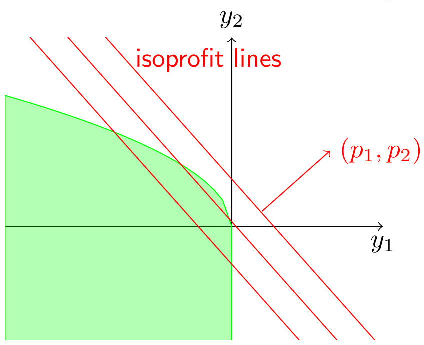
$$y, y' \in Y \implies \alpha y + (1 - \alpha) y' \in Y.$$

2.1.1 The problem of profit maximization

PMP :

$$\max p \cdot y \qquad \text{subject to } y \in Y;$$

it is implicitly assumed, as we did when talking about consumers, that the firm is a *price-taker*: we are not talking about Microsoft, or any monopolist firm, which can *choose*, or at least have influence on, p.



$Y(p)$, the *set of solutions* to PMP, will not necessarily be a singleton.
Given any $y \in Y(p)$, we will have that the profit will be just py.

02/02/2011

Let's consider a special case of the profit maximization problem: having only 1 output.

We will have a production function

$$f(z) = q, \qquad z \in \mathbb{R}_+^{L-1}$$

and we want to solve

$$\max_{z \geq 0} \; p \cdot f(z) - wz$$

where w, wages (and costs of inputs in general), are denoted separately. p, the price of the output, is a scalar.

The first order condition comes out

$$p \frac{\partial f(z^*)}{\partial z_l} - w_l \geq 0$$

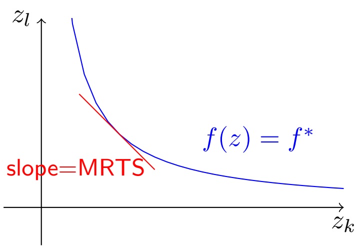
where we will have

$$w = 0 \iff z_l > 0 \qquad \forall i = 1, 2, L-1.$$

If $z_l^* > 0$ and $z_k^* > 0$, the above first order condition gives:

$$\frac{\frac{\partial f(z^*)}{\partial z_l}}{\frac{\partial f(z^*)}{\partial z_k}} = \frac{w_l}{w_k} \tag{1}$$

which is the *marginal rate of technical substitution*, telling *how much we will have to increase some given input if some other input decreases, in order to leave production unchanged*.



By the way, this definition could be resumed in "take the total differential" (we are applying the implicit function theorem).

Now, suppose that (1) doesn't hold:

$$MRTS_{l,k} < \frac{w_l}{w_k};$$

then, if we give up one unit of input l, we get *more* units of input k (right hand side) than we would need to just leave production unchanged (left hand side). Formally, looking at the change in profits,

$$\begin{aligned}\Delta\Pi =& P\Delta q - w_l \Delta z_l - w_k \Delta z_k \\ =& w_l \Delta z - w_k z_k\end{aligned}$$

and if we set $\Delta z_h = MRTS_{l,h}\Delta_l$,

$$\begin{aligned}\Delta\Pi =& - w_l \Delta z_l - w_k(-MRTS_{l,k})\Delta z_l \\ =& w_k \Delta z_l \left[-\frac{w_l}{w_k} + MRTS_{l,k}\right]\end{aligned}$$

so if indeed $MRTS_{l,k} > \frac{w_l}{w_k}$, we cannot be profit maximizers, since we are not minimizing costs: there is a way to get the same output and pay less costs.

A very important consequence is that if we have two firms j and j' using the same kind of inputs and paying them the same prices, then

$$MRTS^j_{l,k} = MRTS^{j'}_{l,k}$$

even if *they are not coordinating* - the price of the inputs themselves is a sort of communication channel.

Indeed, if we had

$$MRTS^j_{l,k} > MRTS^{j'}_{l,k}$$

then by just exchanging an unit of input l with $MRTS^j_{l,k}$ units of input k from firm j', firm j would keep the production unchanged, the same would hold form firm j', and the system of the two firms would be saving some amount of inputs. So, the situation would not be efficient ("efficiency" $\implies$ you cannot produce the same output using less inputs).

So the final lesson is that in a perfect market economy, prices act as signals for millions of individuals which, coordinating, do something that is far more complex than what a central planner could afford.[5]

So far we have $\Pi(p)$, the value of PMP, and $y(P)$, the solution to PMP. What happens if we take $\Pi(\alpha p)$? All prices - of inputs *and* outputs - are doubled, but the optimal choice of inputs and outputs will *not* change, and hence the firm's gains will *increase* by α. Formally, Π is *homogeneous of degree 1* in its arguments, while y is homogeneous of degree 0.

If Y is convex, so is $y(p)$: indeed, given $y, y' \in y(p)$, then

$$py = py' = \Pi(p)$$

[5]OK, I'm just *taking notes* and not *expressing my point of view*, clear? Don't try this at home.

and

$$p(\alpha y + (1-\alpha)y') = py = \Pi(p).$$

If moreover Y is *strictly* convex, $y(p)$ is a singleton. The reason is the same that we saw when discussing utility maximization.

Theorem 16 (Law of supply)**.**

$$[p' - p]\,[y' - y] \geq 0 \qquad \forall y \in y(p), y' \in y(p').$$

Proof.

$$p'[y' - y] = p'y' - p'y \geq 0$$

and

$$p[y' - y] \leq 0$$

so

$$p'[y' - y] - p[y' - y] \geq 0.$$

□

What is the interpretation? If the price of output l increases, the the production of output l cannot *decrease*. If a price of input l increases, the *use* of that output cannot *increase*.

2.2 Cost minimization

Let's consider the case of only one output: profit maximization implies cost minimization *but not vice versa!*

$$\min_{z \geq 0} w \cdot z \qquad \text{subject to } f(z) = q$$

If we want to obtain profit maximization, we need to steps: the first is indeed costs minimization, the second is

$$\max_{w} p \cdot q - c(w, q)$$

with solution

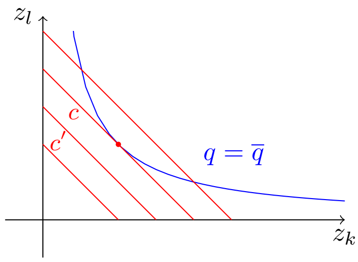
$$p - \frac{\partial C}{\partial q} \geq 0$$

and in particular

$$p - \frac{\partial C}{\partial q} = 0 \text{ if } q > 0$$

An interesting thing is that the cost minimization problem holds even when profit maximization is not possible: for instance, when a monopolist firm chooses its prices, it is still minimizing costs.

The result of the costs minimization problem is the *conditional demand*, $z(w, q)$, which tells the optimal amount of each good given prices and some production level.



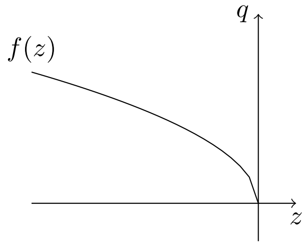
By the way,

$$c(\alpha w, q) = \alpha c(w, q)$$

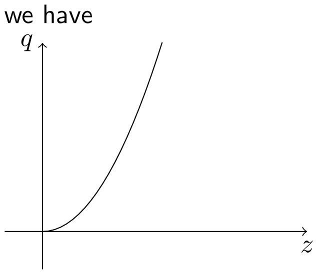
while $z(\cdot, q)$ is instead homogeneous of degree 0 in w.

Moreover, if we have constant returns to scale, then $c(w, \cdot)$ is homogeneous of degree 1 in q. If we have *non-increasing* returns to scale, c is convex in q: intuitively, to double output, we need to *more* than the double the inputs.

For instance, let's take the example of 1 input, 1 output:



we have



Now, if we consider

$$\max_{q \geq 0} \; pq - c(w, q)$$

we get the first order condition

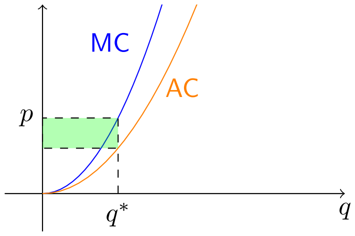
$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0$$

(with equality if $q > 0$).

We can rewrite this in terms of *marginal cost*:

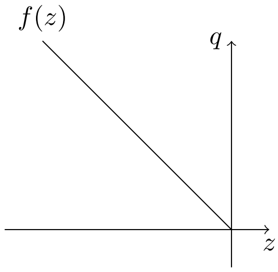
$$p - MC(q^*) \leq 0$$

and marginal cost precisely gives us the supply curve:

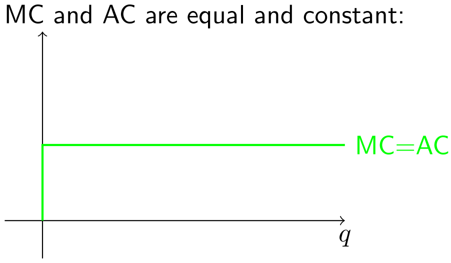


while instead the *average costs* always lies below. Since the optimal q^* is such that the corresponding marginal cost is equal to the price p, the profit of the firm is then precisely the green rectangle.

In case of constant returns to scale,



MC and AC are equal and constant:



and then if $p > MC$, there is *no solution* to firms' problem, and in equilibrium, firms always have *zero profits*. In other terms, if real firms had constant returns to scale and no financial constraints, any given firm would scale as far as it is possible (that is: as long as the assumption that it acts as a price taker is valid).

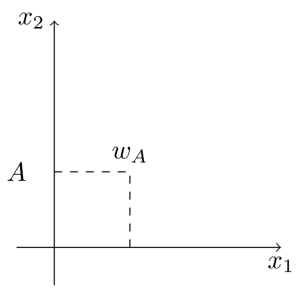
By the way, absence of profits seems slightly less unrealistic if we consider that the wages and even the revenue for the owner(s) can be considered an *implicit* cost, paid for a particular form of input.

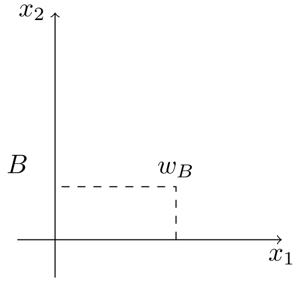
2.3 Efficiency in production

We said that when two firms operate in a market, they are *jointly*, not just individually, maximizing profits: they are *jointly efficient*. It is a way to formalize the idea, which goes back to Adam Smith, that efficient markets will not waste resources.

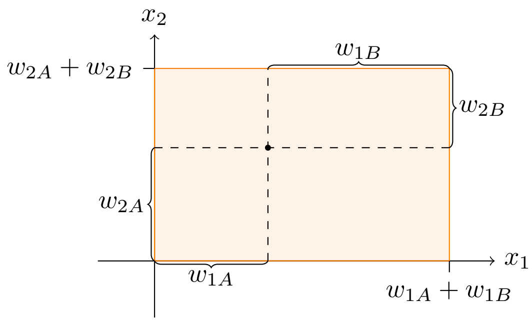
2.4 The Edgeworth box

The Edgeworth box is a way to formalize the idea that if we have two consumers with two commodities, with no production (only an *exchange economy*),





then the Edgeworth box is



An equilibrium in this economy is

$$(p_1, p_2) \text{ and } x_A^*, x_B^*$$

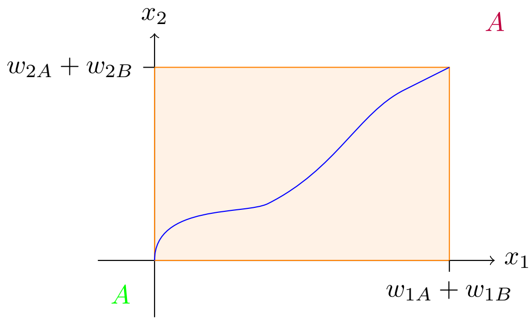
such that

$$x_A^* + x_B^* = w_{2A} + w_{2B}$$

and x_A^*, x_B^* respectively maximize $\succeq_A$ and $\succeq_B$ on (respectively)

$$B^A = \{x | px \leq pw_A\},$$
$$B^B = \{x | px \leq pw_B\}.$$

This is a *Walrasian* equilibrium, and we call it *general* because it holds for all (2, in this case) markets at the same time. Graphically, we can represent equilibria as follows:



Then, the stronger concept of *fair allocation* is sometimes introduced: it requires that

$$x_A^* \succeq_A x_B^*$$

and

$$x_B^* \succeq_B x_A^*,$$

but that is not required in order to characterize *voluntariety* of exchanges.

2.5 Pareto efficiency

A allocation is *Pareto efficient*, or *Pareto optimal*, if

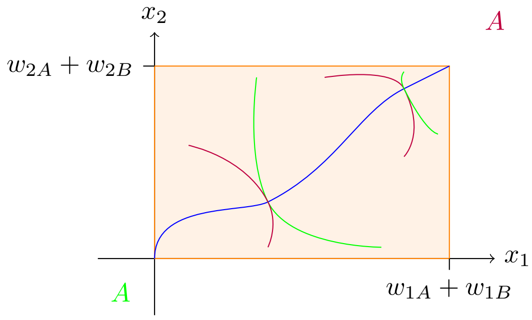
$$(x_A^*, x_B^*) = w_A + w_B$$

(it is feasible), and there is no feasible (x'_A, x'_B) such that

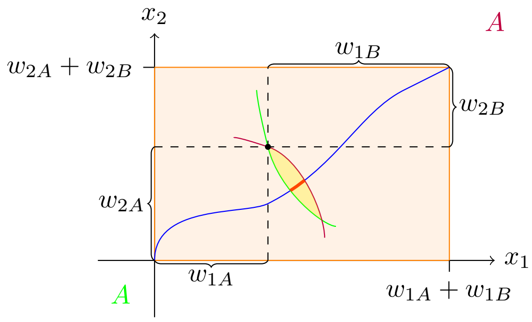
$$x'_A \succeq_A x_A^* \land x'_B \succeq_B x_B^*$$

with strict preferences holding for at least one of the two inequivalences.

In our Edgeworth box above, Pareto efficient allocations are all those in which indifference curves of the two agents are tangent to each other.



and the part of the graph in which both individuals improve their situation with respect to the initial endowments is called the *contract curve*:



In fact, any given Walrasian equilibrium will be on this contract curve, and in general, given any number of consumers and commodities,

Theorem 17 (First Fundamental Theorem of Welfare Economics)**.** *a Walrasian (or* general*, or* competitive*) equilibrium is Pareto efficient.*

It is easy to intuitively understand though many Pareto efficient allocations are *not* what we would like to see in reality, what we would like to see is almost surely a Pareto efficient allocation.